JOURNAL OF APPROXIMATION THEORY 45, 90–98 (1985)

On a Theorem of Deutsch and Kenderov

GERALD BEER

Department of Mathematics,
California State University, Los Angeles, California 90032, U.S.A

Communicated by E. W. Cheney

Received June 3, 1983

1. INTRODUCTION

Let X be a topological space and let Y be a metric space with metric d. Let 2^Y denote the collection of all nonempty subsets of Y. By a *multifunction* from X to Y we mean a function $\Gamma: X \to 2^Y$. A multifunction Γ is called *lower semicontinuous* (l.s.c) if for each open set G in Y $\{x: \Gamma(x) \cap G \neq \emptyset\}$ is an open subset of X. A single valued function $f: X \to Y$ is called a *selection* for Γ if for each $x \in X$ $f(x) \in \Gamma(x)$. Perhaps the most well-known result on the existence of continuous selections is the following theorem of Michael [8]: if X is paracompact and Y is a Banach space and $\Gamma: X \to 2^Y$ is l.s.c. and has closed convex values, then Γ admits a continuous selection.

Michael obtained this result after first proving a more generally applicable approximate selection result. If E is a nonempty subset of Y and $\varepsilon > 0$, let $S_\varepsilon[E]$ denote the union of the open balls in Y whose centers run over E. A function $f: X \to Y$ is called an *ε-approximate selection* for $\Gamma: X \to 2^Y$ if for each x in X $f(x) \in S_\varepsilon[\Gamma(x)]$. Specifically, Michael proved that if X is paracompact and Y is a normed linear space and $\Gamma: X \to 2^Y$ is l.s.c. and has convex values, then for each $\varepsilon > 0$ Γ admits a continuous ε-approximate selection. It is easy to see that lower semicontinuity is not necessary for either of the two above results. Recently, Deutsch and Kenderov [5] characterized those multifunctions defined on a paracompact space with convex values in a normed linear space that admit for each ε a continuous ε-approximate selection as those that are *almost lower semicontinuous* (a.l.s.c.): for each $\varepsilon > 0$ and for each x in X there exists a neighborhood V of x such that $\bigcap \{S_\varepsilon[\Gamma(w)]: w \in V\} \neq \emptyset$. It is easy to see that lower semicontinuity implies almost lower semicontinuity and that almost lower semicontinuity is necessary for the existence of a continuous selection. Following the method of Michael they were able to prove the following result.

0021-9045/85 $3.00

THEOREM. *Let X be paracompact and let Y be a 1-dimensional normed linear space. Suppose $\Gamma: X \to 2^Y$ has compact convex values. Then Γ admits a continuous selection if and only if Γ is a.l.s.c..*

Naturally, they asked if this result held more generally. In this article we show that their result is best possible, i.e., it fails if Y is 2 dimensional. We also present several characterizations of closed convex valued multifunctions with values in a Banach space that admit continuous selections in terms of the notion of almost lower semicontinuity, one of which involves the existence of a fixed point for a certain map on the submultifunctions of the given one. Finally, we prove some selection and approximate selection theorems for nonconvex valued a.l.s.c. multifunctions.

Before proceeding we present some additional notation and terminology. If A is a subset of a topological space, $\bar{A}$ will denote the closure of A; if A is a subset of a linear space, $\text{conv}(A)$ will denote the convex hull of A. If X and Y are topological spaces, a function $f: X \to Y$ is said to be of *Baire class* $\alpha < \Omega$ if for each open set G in Y $f^{-1}(G)$ is a set of additive class α in X. In particular $f: X \to Y$ is said to be of *Baire class one* (resp. *two*) if for each open set G in Y $f^{-1}(G)$ is an F_σ (resp. $G_{\delta\sigma}$) set. For a thorough discussion of such functions the reader should consult [6], where the functions of Baire class α are called B-measurable of class α. Suppose now that Y is a metric space and $\Gamma: X \to 2^Y$ is a multifunction. If $E \subset Y$ we write $\Gamma^{-1}(E)$ for the set $\{x: \Gamma(x) \cap E \neq \varnothing\}$. If for each $n \in Z^+$ $\Gamma_n: X \to 2^Y$, we will say that $\langle \Gamma_n \rangle$ *converges to* Γ if for each $x \in X$ $\langle \Gamma_n(x) \rangle$ converges to $\Gamma(x)$ in *Hausdorff distance* [3]: for each $\varepsilon > 0$ there exists $N \in Z^+$ such that for each $n \geqslant N$ both $S_\varepsilon[\Gamma(x)] \supset \Gamma_n(x)$ and $S_\varepsilon[\Gamma_n(x)] \supset \Gamma(x)$. This notion, as well as more general notions of convergence of multifunctions (and the convergence of associated measurable selections), is considered in a recent paper of Salinetti and Wets [10].

Let $\varepsilon > 0$ and again let $\Gamma: X \to 2^Y$. For each $x \in X$ define $\Gamma(\varepsilon; x) \subset Y$ as follows:

$$\Gamma(\varepsilon; x) = \{ y: \text{for some neighborhood } V \text{ of } x$$
$$y \in \bigcap \{S_\varepsilon[\Gamma(w)]: w \in V\}\}.$$

Clearly, Γ is a.l.s.c. if and only if for each $\varepsilon > 0$ and x in X the set $\Gamma(\varepsilon; x)$ is nonempty. For each x let $\theta_\Gamma(x) = \bigcap_{\varepsilon > 0} \Gamma(\varepsilon; x)$. Evidently, $\theta_\Gamma(x) \subset \overline{\Gamma(x)}$, and if Γ admits a continuous selection f then $f(x) \in \theta_\Gamma(x)$. Example 2 below shows that almost lower semicontinuity if Γ does not ensure that the sets $\{\theta_\Gamma(x): x \in X\}$ are nonempty. However, we shall see that this is the case if Γ is a.l.s.c. and compact valued.

2. Continuous Selections for Convex Valued Multifunctions

We first produce an a.l.s.c. multifunction Γ: $[0, 1] \to 2^{R \times R}$ with compact convex values that fails to admit a continuous selection.

Example 1. For each $n \in Z^+$ let $a_n = \frac{1}{2}[1/n + 1/(n+1)]$. Define Γ: $[0, 1] \to 2^{R \times R}$ as follows:

$$\Gamma(x) = (0, 0) \qquad \text{if } x = 0$$
$$= \operatorname{conv}\left[\left(\frac{1}{n}, 1\right), (a_n, 0), \left(\frac{1}{n+1}, 1\right)\right] \quad \text{if } \frac{1}{n+1} < x \leqslant \frac{1}{n}.$$

The values of Γ are all compact convex sets. It is easy to see that

$$\theta_\Gamma(x) = \Gamma(x) \qquad \text{if} \quad x \notin \{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}$$
$$= \{(1/n, 1)\} \qquad \text{if} \quad x = 1/n \text{ for some } n > 1.$$

Since for each $x \in [0, 1]$ the set $\theta_\Gamma(x)$ is nonempty, we conclude that $\Gamma(\varepsilon; x)$ is nonempty for each x and $\varepsilon > 0$, whence Γ is a.l.s.c. Now if f were a continuous selection for Γ, then the requirement $f(x) \in \theta_\Gamma(x)$ would imply (i) $f(0) = (0, 0)$ and (ii) for each integer $n > 1$ $f(1/n) = (1/n, 1)$. Since this is incompatible with continuity, no continuous selection for Γ exists.

We next show that an a.l.s.c. multifunction Γ: $[0, 1] \to 2^{R \times R}$ with closed convex values need not admit a Borel measurable selection.

Example 2. Let B be non-Borel subset of $[0, 1]$. Define Γ: $[0, 1] \to 2^{R \times R}$ by

$$\Gamma(x) = \{(y, z) : y > 0, z \geqslant 1/y\} \qquad \text{if} \quad x \in B$$
$$= \{(0, z) : z \geqslant 0\} \qquad \text{if} \quad x \notin B.$$

Let n be an arbitrary positive integer. Then for each $x \in [0, 1]$ there exists $(y, z) \in \Gamma(x)$ such that $\|(y, z) - (0, n)\| \leqslant 1/n$: if $x \in B$ take $(y, z) = (1/n, n)$, and if $x \notin B$, take $(y, z) = (0, n)$. Thus, Γ is a.l.s.c. But if f is a selection for Γ, we have

$$f^{-1}(\{y, z) : y > 0, z \geqslant 1/y\}) = B.$$

Thus, the inverse image under f of a closed set need not be Borel, whence f is not Borel measurable.

The last example shows that if X is paracompact and Y is a Banach space and Γ: $X \to 2^Y$ is a.l.s.c. and has closed convex values, then Γ need not admit even a Borel selection. Still, we can characterize those Γ that admit continuous slections in terms of the maps $\Gamma_1 \to \Gamma_1(\varepsilon; \cdot)$ and $\Gamma_1 \to \theta_{\Gamma_1}$ defined on submultifunctions of Γ. We need a preliminary lemma. It should be noted that both this lemma and Lemma 2 below are much in the spirit of the more general results of Banzaru [1].

LEMMA 1. *Let X be a topological space and let Y be a metric space. Suppose $\Gamma: X \to 2^Y$ has closed values. Consider the following statements*:

(1) *Γ is a.l.s.c.*

(2) $\theta_\Gamma = \Gamma$.

(3) *Γ is l.s.c.*

(4) *θ_Γ is l.s.c.*

(5) *$\langle \Gamma(1/n; \cdot) \rangle$ converges locally uniformly to θ_Γ.*

Then

(a) *statements* (2) *and* (3) *are equivalent and thus imply* (4);

(b) *statement* (4) *implies* (1);

(c) *statement* (5) *implies* (4).

Proof. (a) Assume first that Γ is l.s.c. Since for each x $\Gamma(x)$ is a closed set we have $\theta_\Gamma(x) \subset \Gamma(x)$. To show the reverse inclusion let $y \in \Gamma(x)$ and let ε be positive. Since $S_\varepsilon[y]$ is open and Γ is l.s.c., there is a neighborhood V of x such that $V \subset \Gamma^{-1}(S_\varepsilon[y])$. Thus, $y \in \bigcap \{S_\varepsilon[\Gamma(w)]: w \in V\}$ so that $y \in \Gamma(\varepsilon; x)$. Since y and $\varepsilon > 0$ were arbitrary we have $\Gamma(x) \subset \theta_\Gamma(x)$. Conversely, suppose $\Gamma = \theta_\Gamma$ and $\Gamma(x)$ meets some open set G in Y. Choose $y \in \Gamma(x)$ and $\varepsilon > 0$ such that $S_\varepsilon[y] \subset G$. Since $y \in \Gamma(\varepsilon; x)$ there exists a neighborhood V of x such that $y \in S_\varepsilon[\Gamma(w)]$ for each w in V. Thus, for each such w $\Gamma(w) \cap G \neq \varnothing$, whence Γ is l.s.c.

(b) The multifunction θ_Γ must then be a.l.s.c., and each multifunction that contains an a.l.s.c. multifunction must itself be a.l.s.c.

(c) Let $x \in X$ and let V be a neighborhood of x on which $\langle \Gamma(1/n; \cdot) \rangle$ converges uniformly to θ_Γ. Let $G \subset Y$ be open, and suppose $y \in \theta_\Gamma(x) \cap G$. Choose $\varepsilon > 0$ for which $S_\varepsilon[y] \subset G$. Pick $n \in Z^+$ so large that $1/n < \varepsilon$ and for each $w \in V$

$$\Gamma(1/n; w) \subset S_{\varepsilon/2}[\theta_\Gamma(w)].$$

Since $y \in \Gamma(1/2n; x)$ there exists a neighborhood W of x contained in V and for each $w \in W$ a point $y_w \in \Gamma(w)$ for which $d(y, y_w) < 1/2n$. It follows that $y_w \in \Gamma(1/n; w)$, whence by the choice of n there exists y'_w in $\theta_\Gamma(w)$ for which

$d(y_w, y'_w) < \varepsilon/2$. Thus, for each $w \in W$ we have $y'_w \in S_\varepsilon[y] \subset G$, and θ_Γ is l.s.c. at x.

THEOREM 1. *Let X be a paracompact space and let Y be a Banach space. Let $\Gamma: X \to 2^Y$ be an a.l.s.c. multifunction with closed convex values. Consider the following statements.*

(1) *Γ admits a continuous selection f.*

(2) *Γ contains a l.s.c. multifunction Γ_1.*

(3) *Γ contains a multifunction Γ_1 for which $\Gamma_1 = \theta_{\Gamma_1}$.*

(4) *Γ contains a multifunction Γ_1 for which $\langle \Gamma_1(1/n; \cdot)\rangle$ converges locally uniformly to θ_{Γ_1}.*

Then

(a) *statements* (1) *through* (3) *are equivalent;*

(b) *statement* (4) *implies all the others;*

(c) *if X is locally compact, then statements* (1) *through* (4) *are equivalent.*

Proof. (a) Clearly (1) implies (3): take $\Gamma_1 = f$. The previous lemma yields (3) implies (2). If (2) holds then the multifunction $x \to \overline{\text{conv}(\Gamma_1(x))}$ is a l.s.c. multifunction with closed convex values contained in Γ. Hence, by Michael's theorem it has a continuous selection contained in Γ.

(b) By Lemma 1 above, (4) implies (2) and hence all the others.

(c) Since (a) and (b) are established, it suffices to prove that (1) implies (4). We assume (1) holds and take $\Gamma_1 = f$. Fix x in X and let K be a compact neighborhood of x. By the continuity of f, for each $\varepsilon > 0$ and w in K we have $\Gamma_1(\varepsilon; w) = S_\varepsilon[f(w)]$. Now Y is a linear space; so, for each positive δ we have $S_\delta[S_\varepsilon[f(w)]] = S_{\varepsilon+\delta}[f(w)]$. From this fact and the continuity of f, for each w in K there is a neighborhood V_w of w such that for each z in V_w we have $\Gamma_1(\varepsilon; z) \subset S_\delta[\Gamma_1(\varepsilon; w)]$. Since Γ_1 is l.s.c. and compact valued and the sequence $\langle \Gamma(1/n; \cdot)\rangle$ converges to Γ_1 on K, by a Dini-type theorem for multifunctions (see, e.g., Theorem 3 of [2]), the convergence must be uniform to $\Gamma_1 = \theta_{\Gamma_1}$ on K.

3. SELECTIONS AND APPROXIMATE SELECTIONS FOR NONCONVEX MULTIFUNCTIONS

By Example 2 there is no hope of showing that each closed valued a.l.s.c. multifunction defined on a metric space X with values in a complete metric space Y admits a Borel selection. We shall show, however, that each compact valued a.l.s.c. multifunction with values in a separable metric space admits a Borel selection; in fact, it must admit a Baire class two selection. Our proof rests on the following version of the Kuratowski–Ryll-Nardzewski selection theorem [7].

KRN SELECTION THEOREM. *Let X be a metric space and let Y be a separable complete metric space. Suppose $\Gamma: X \to 2^Y$ has closed values, and for each open subset G of Y $\Gamma^{-1}(G)$ is of additive class α. Then Γ admits a Baire class α selection.*

Although a compact valued a.l.s.c. multifunction Γ need not be "Borel measurable," we shall show that its auxiliary multifunction θ_Γ has the following property: for each open subset G of Y $\theta_\Gamma^{-1}(G)$ is a $G_{\delta\sigma}$ subset of X. Of course, we first need to show that for each x the set $\theta_\Gamma(x)$ is nonempty.

LEMMA 2. *Let X be a topological space and let Y be a metric space. Suppose $\Gamma: X \to 2^Y$ is a.l.s.c. and compact valued. Then for each x in X the set $\theta_\Gamma(x)$ is a nonempty compact subset of $\Gamma(x)$, and $\langle \Gamma(1/n;\cdot)\rangle$ converges to θ_Γ on X.*

Proof. We first establish (∗): for each x in X whenever $\{\varepsilon_n\}$ is a sequence of positive numbers convergent to zero and for each $n \in Z^+$ $y_n \in \Gamma(\varepsilon_n; x)$, then $\langle y_n \rangle$ has a subsequence convergent to some point y of $\theta_\Gamma(x)$. For each n choose a point y'_n in $\Gamma(x)$ for which $d(y_n, y'_n) < \varepsilon_n$. By passing to a subsequence we can assume $\langle y'_n \rangle$ converges to a point $y \in \Gamma(x)$. Now for each n $y'_n \in \Gamma(2\varepsilon_n; x)$. Hence, if for each n we set $\lambda_n = d(y'_n, y)$, we have $y \in \Gamma(2\varepsilon_n + \lambda_n; x)$. Thus $y \in \theta_\Gamma(x)$. Property (∗) immediately implies that $\theta_\Gamma(x)$ is a closed set. Since $\Gamma(x)$ is closed, we also have $\theta_\Gamma(x) \subset \Gamma(x)$, whence $\theta_\Gamma(x)$ is a nonempty compact set. Now let $\lambda > 0$. If no $n \in Z^+$ exists such that $\Gamma(1/n; x) \subset S_\lambda[\theta_\Gamma(x)]$, then invoking (∗) once again, $\theta_\Gamma(x) - S_\lambda[\theta_\Gamma(x)]$ would be nonempty, an impossibility. Thus for each $\lambda > 0$ there exists $N \in Z^+$ such that for each $n \geqslant N$ $\Gamma(1/n; x) \subset S_\lambda[\theta_\Gamma(x)]$. We always have $\theta_\Gamma(x) \subset S_\lambda[\Gamma(1/n; x)]$, and the convergence of $\langle \Gamma(1/n; \cdot)\rangle$ to θ_Γ is established.

Our next lemma implies that if Γ is a.l.s.c., then for each $\varepsilon > 0$ the auxiliary multifunction $\Gamma(\varepsilon; \cdot)$ is l.s.c. One can also easily show that if Γ is convex valued, the same can be said for each auxiliary multifunction.

LEMMA 3. *Let X be a topological space, let Y be a metric space, and let $\Gamma: X \to 2^Y$ be a.l.s.c. Then for each $\varepsilon > 0$ and for each subset E of Y, $\Gamma(\varepsilon; \cdot)^{-1}(E)$ is open.*

Proof. Suppose $\Gamma(\varepsilon; x) \cap E \neq \varnothing$. Select y in the intersection; then there exists a neighborhood V of x such that $y \in \bigcap \{S_\varepsilon[\Gamma(w)]: w \in V\}$. But since V is a neighborhood of each point w in V, we have $y \in \Gamma(\varepsilon; w)$ for each such point. Thus $V \subset \Gamma(\varepsilon; \cdot)^{-1}(E)$.

THEOREM 2. *Let X be a metric space and let Y be a separable metric space. Suppose $\Gamma: X \to 2^Y$ is a compact valued a.l.s.c. multifunction. Then Γ admits a Baire class two selection.*

Proof. By Lemma 2 for each x the set $\theta_\Gamma(x)$ is nonempty. Let $G \subset Y$ be an open set; we claim that $\theta_\Gamma^{-1}(G)$ is a $G_{\delta\sigma}$ subset of X. For each $n \in Z^+$ let $B_n = (S_{1/n}[G^c])^c$; then B_n is a closed set, $B_n \subset B_{n+1}$, and $G = \bigcup_{n=1}^\infty B_n$. Let $E \subset X$ be defined by

$$E = \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty \bigcap_{j \geq k} \left\{x: \Gamma\left(\frac{1}{j}; x\right) \cap B_n \neq \varnothing\right\}.$$

We first show that $\theta_\Gamma^{-1}(G) = E$. Let $x \in E$; then for some integers k and n

$$\Gamma(1/j; x) \cap B_n \neq \varnothing \qquad (j = k, k+1, \ldots).$$

For each $j \geq k$ choose y_j in the intersection. The property $(*)$ established in the proof of Lemma 2 implies that $\langle y_j \rangle$ has a subsequence convergent to some point y in $\theta_\Gamma(x)$. Since B_n is closed, $y \in B_n \subset G$. Hence $x \in \theta_\Gamma^{-1}(G)$. On the other hand if $\theta_\Gamma(x) \cap G \neq \varnothing$, then for some n $\theta_\Gamma(x) \cap B_n \neq \varnothing$ whence

$$x \in \bigcap_{j=1}^\infty \left\{w: \Gamma\left(\frac{1}{j}; w\right) \cap B_n \neq \varnothing\right\}.$$

It follows that $x \in E$. Thus, by Lemma 3, for each open subset G of Y the set $\theta_\Gamma^{-1}(G)$ is a $G_{\delta\sigma}$ subset of X. Viewing θ_Γ as a multifunction from X to the completion of Y, the KRN selection theorem yields a Baire class two selection for θ_Γ which is, a fortiori, a selection for Γ.

Čoban [4] has shown that if X is metric and Y is metric (resp. complete metric) and $\Gamma: X \to 2^Y$ is l.s.c. with compact (resp. closed) values, then Γ admits a Baire class one selection. QUESTION: Does an a.l.s.c. compact valued multifunction with metric domain and codomain admit a Baire class one selection?

If X and Y are metric, a compact valued l.s.c. multifunction $\Gamma: X \to 2^Y$ need not admit for each $\varepsilon > 0$ a continuous ε-approximate selection. For example, if $\Gamma: [0, 1] \to 2^R$ is defined by

$$\begin{aligned} \Gamma(x) &= \{1/x\} && \text{if } \tfrac{1}{4} \leq x < 1 \\ &= \{1, 1/x\} && \text{if } 0 < x < \tfrac{1}{4} \\ &= \{1\} && \text{if } x = 0 \end{aligned}$$

then Γ does not admit a continuous 1-approximate selection. Thus, if X and Y are metric and $\Gamma: X \to 2^Y$ is a.l.s.c., then Γ need not admit for each $\varepsilon > 0$ a continuous ε-approximate selection. Our final result is, therefore, best possible.

THEOREM 3. *Let X and Y be metric spaces and let $\Gamma: X \to 2^Y$ be a.l.s.c. Then for each $\varepsilon > 0$ there exists a Baire class one function $f: X \to Y$ such that for each x in X $f(x) \in S_\varepsilon[\Gamma(x)]$.*

Proof. For each x in X there exists a neighborhood V_x of x such that $\bigcap \{S_\varepsilon[\Gamma(w)]: w \in V_x\}$ is nonempty. For each x pick a point $y(x)$ in the intersection. Since X is paracompact and regular there exists a locally finite open cover $\{U_i: i \in I\}$ such that $\{\bar{U}_i: i \in I\}$ refines $\{V_x: x \in X\}$. For each $i \in I$ pick $x(i) \in X$ such that $\bar{U}_i \subset V_{x(i)}$. Next, well order Y and define $f: X \to Y$ by

$$f(w) = \min\{y(x(i)): w \in \bar{U}_i\}.$$

We first show that f is of Baire class one. Since relatively F_σ subsets of each open subset of X are themselves F_σ subsets of X, by a theorem of Montgomery [9], it suffices to show that f is locally of Baire class one. Fix z in X and pick an open neighborhood W of z that meets only finitely many members of the closed cover, say, $\bar{U}_{i_1}, \bar{U}_{i_2}, \dots, \bar{U}_{i_n}$. Now write $f(W)$, a subset of the set $\{y(x(i_m)): m = 1, \dots, n\}$, in increasing order, say, $\{y_1, y_2, \dots, y_p\}$, where $p \leqslant n$. We claim that for each $j \leqslant p$ $(f \mid W)^{-1}(\{y_1, y_2, \dots, y_j\})$ is a relatively closed subset of W. To see this let $\langle w_k \rangle$ be a sequence in the inverse image convergent to a point w in W. There exist $l \in \{1, 2, \dots, j\}$ such that y_l is a value of $f(w_k)$ infinitely often. Thus, for some $m \in \{1, 2, \dots, n\}$ $w_k \in \bar{U}_{i_m}$ and $y(x(i_m)) = y_l$ for infinitely many indices k. It follows that $w \in \bar{U}_{i_m}$, whence $f(w) \leqslant y_l$, i.e., $w \in f^{-1}(\{y_1, y_2, \dots, y_j\})$. This establishes the claim. Now for each $j \in \{1, \dots, p\}$

$$(f \mid W)^{-1}(y_j) = (f \mid W)^{-1}(\{y_1, \dots, y_j\}) - (f \mid W)^{-1}(\{y_1, \dots, y_{j-1}\})$$

is an F_σ set, whence for each open set G in Y

$$(f \mid W)^{-1}(G) = (f \mid W)^{-1}(G \cap \{y_1, \dots, y_p\})$$

is an F_σ set. Thus $f \mid W$ is of Baire class one; so, f is globally of Baire class one.

To see that f is an ε-approximate selection for Γ, again fix z in X. By the definition of f there exists $x \in X$ such that $z \in V_x$ and $f(z) = y(x)$. However, $y(x) \in S_\varepsilon[\Gamma(w)]$ for each w in V_x; so, in particular, $f(z) \in S_\varepsilon[\Gamma(z)]$.

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